

# Torus Quotients as Global Quotients by Finite Groups

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## Abstract

This article is motivated by the following local-to-global question: is every variety with tame quotient singularities *globally* the quotient of a smooth variety by a finite group? We show that this question has a positive answer for all quasi-projective varieties which are expressible as a quotient of a smooth variety by a split torus (e.g. simplicial toric varieties). Although simplicial toric varieties are rarely *toric* quotients of smooth varieties by finite groups, we give an explicit procedure for constructing the quotient structure using toric techniques.

This result follows from a characterization of varieties which are expressible as the quotient of a smooth variety by a split torus. As an additional application of this characterization, we show that a variety with *abelian* quotient singularities may fail to be a quotient of a smooth variety by a finite *abelian* group. Concretely, we show that  $\mathbb{P}^2/A_5$  is not expressible as a quotient of a smooth variety by a finite abelian group.

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The second author was supported by NSF grant DMS-0943832 and an NSF postdoctoral fellowship (DMS-1103788).

2010 *Mathematics Subject Classification*. 14L30, 14D23, 14M25.

# 1 Introduction

In this paper, we investigate a local-to-global question concerning quotient singularities. Recall that a variety  $X$  over a field  $k$  is said to have tame quotient singularities if it is étale locally the quotient of a smooth variety by a finite group whose order is prime to the characteristic.<sup>1</sup> Every variety of the form  $U/G$ , where  $U$  is smooth and  $G$  is a finite group of order relatively prime to the characteristic of  $k$  must clearly have at worst tame quotient singularities. The motivation for this paper is whether the converse is true, a question suggested to us by William Fulton:

**Question 1.** *Let  $k$  be an algebraically closed field. If  $X$  is a variety over  $k$  with tame quotient singularities, then does there exist a smooth variety  $U$  over  $k$  with an action of a finite group  $G$  such that  $X = U/G$ ?*

While we do not answer Question 1 in general, we show that it has an affirmative answer when  $X$  is quasi-projective and globally a quotient of a smooth variety by a torus. In particular, every quasi-projective simplicial toric variety (with tame singularities) is a global quotient by a finite abelian group (see Corollary 2.13). These results are immediate consequences of the following characterization of quotients of smooth varieties by finite abelian groups. Recall that  $X$  is said to have *abelian* quotient singularities if it is étale locally the quotient of a smooth variety by a finite abelian group.

**Theorem 1.1.** *Let  $X$  be a quasi-projective variety with tame abelian quotient singularities over an algebraically closed field  $k$ . The following are equivalent.*

1.  *$X$  is a quotient of a smooth quasi-projective variety by a finite abelian group.*
2.  *$X$  is a quotient of a smooth quasi-projective variety by a torus acting with finite stabilizers.*
3.  *$X$  has Weil divisors  $D_1, \dots, D_r$  whose images generate  $\text{Cl}(\widehat{\mathcal{O}}_{X,x})$  for all points  $x$  of  $X$ .*
4. *The canonical stack over  $X$  is a stack quotient of a quasi-projective variety by a torus (see Remark 1.3).*

*Remark 1.2.* Theorem 1.1 is a consequence of the more technical Theorem 5.1, which applies to algebraic spaces over infinite fields.

*Remark 1.3* (For non-stack-theorists). For those not familiar with canonical stacks, (4) says that  $X$  can be expressed as a quotient of a smooth variety  $U$  by a torus  $T$  which acts freely on the preimage of the smooth locus of  $X$ . In fact, this implies that the stabilizers of the  $T$ -action are as small as one could hope for (in a precise sense) over the singular locus of  $X$  as well.

*Remark 1.4* (For stack-theorists). Those familiar with stacks should not confuse Question 1 with the question “is every smooth tame Deligne-Mumford stack the *stack quotient* of a smooth scheme by a finite group?” The answer to this question is no, e.g. the weighted projective stack  $\mathbf{P}(1, 2)$ . Question 1 allows for the possibility of extra ramification over the coarse space. The appropriate stacky generalization of Question 1 is, “is every smooth tame Deligne-Mumford stack  $\mathcal{X}$  a *relative coarse space* of a stack quotient of a smooth scheme by a finite group?” Our proof of (2)  $\Rightarrow$  (1) shows that the answer to this question is affirmative when  $\mathcal{X}$  is a stack quotient by a split torus and has quasi-projective coarse space.

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<sup>1</sup>Alternatively,  $X$  has tame quotient singularities if all complete local rings  $\widehat{\mathcal{O}}_{X,x}$  are isomorphic to the ring of invariants in  $k(x)[[t_1, \dots, t_n]]$  under the action of a finite group of order relatively prime to the characteristic of  $k$ .

We would like to emphasize that even when  $X$  is a toric variety, it is not obvious that Question 1 has an affirmative answer. We show in Proposition 4.1 that if  $X$  is the blow-up of weighted projective space  $\mathbb{P}(1, 1, 2)$  at a smooth torus-fixed point, then it is not possible to present  $X$  as a *toric* quotient by a finite group. Nevertheless, we show in §3 that the proof of our main theorem gives an effective toric procedure for constructing  $U$  and  $G$  when  $X$  is a toric variety (see Theorem 3.1). We demonstrate this procedure for the example of  $\mathbb{P}(1, 1, 2)$  blown-up at a smooth torus-fixed point in §4, obtaining it as an explicit quotient of a smooth variety by  $\mathbb{Z}/2$ .

There are several variants of Question 1 that one can pose. For example,

**Question 2.** *If  $X$  is a variety over an algebraically closed field with tame abelian quotient singularities, then is it of the form  $U/G$  with  $U$  a smooth variety and  $G$  a finite abelian group?*

We answer this question negatively with the following result. The key input is the equivalence of (1) and (4) in Theorem 1.1, which shows that if the canonical stack over  $X$  is not *stack quotient* by a torus, then  $X$  cannot be expressed as a *scheme quotient* of a smooth scheme by a finite abelian group.

**Theorem 1.5.** *Let  $k$  be an algebraically closed field with  $\text{char}(k) \nmid 60$ , and let  $V$  be an irreducible 3-dimensional representation of the alternating group  $A_5$ .*

1.  *$X = \mathbb{P}(V)/A_5$  has abelian quotient singularities, but is not a quotient of a smooth variety by a finite abelian group.*
2. *Moreover, if  $Y$  is obtained from  $X$  by resolving any two of its three singularities, then  $Y$  has precisely one abelian quotient singularity and is also not a quotient of a smooth variety by a finite abelian group.*

There are other variants of Question 1 in the literature whose answers are known to be positive. If one modifies Question 1 by dropping the assumption that  $G$  be finite, then there is an affirmative answer. By [EHKV01, Corollary 2.20], if  $X$  is a variety with quotient singularities over a field of characteristic 0, then  $X = U/G$ , where  $U$  is a smooth scheme and  $G$  is a linear algebraic group.

If one modifies Question 1 in a different direction, requiring a finite surjection  $U \rightarrow X$  with  $U$  smooth, but no group action, then the answer is also affirmative. It follows from [KV04, Theorem 1] and [EHKV01, Theorem 2.18] that for an irreducible quasi-projective variety  $X$  with quotient singularities over a field  $k$ , there is a finite surjection from a smooth variety to  $X$ .

Question 1 therefore asks if there is a common refinement of these two variants.

In the appendix, we prove Theorem 6.1, which shows that a smooth Deligne-Mumford stack  $\mathcal{X}$  with finite diagonal and trivial generic stabilizer is determined by its coarse space  $X$  and the ramification divisor of the coarse space morphism  $\mathcal{X} \rightarrow X$ . We use a special case of this result to prove Theorem 1.1, but we believe the general result will be of broad interest.

## Acknowledgments

We would like to thank Dan Abramovich, Dan Edidin, Tom Graber, and David Rydh for helpful conversations. We would especially like to thank Bill Fulton for suggesting Question 1 and the interest he took in this project.

## 2 Answering Question 1 affirmatively for torus quotients

This section is devoted to the proof of Theorem 2.9, which shows that (2) implies (1) in Theorem 1.1 (see Remark 2.10). We begin by collecting some well-known definitions and results.

Given a finitely-generated abelian group  $A$  and a scheme  $S$ , we have a group scheme  $D_S(A) := \text{Spec } \mathcal{O}_S[A]$  over  $S$ . Any group scheme obtained in this way is called *diagonalizable*.  $D_S(A)$  is called *tame* if the order of  $A$  is prime to the characteristic of all residue fields of  $S$ . Note that diagonalizable group schemes are nothing other than group schemes of the form  $\mathbb{G}_m^r \times \mu_{n_1} \times \cdots \times \mu_{n_\ell}$ . In particular, a tame finite diagonalizable group scheme over an algebraically closed field is simply a finite abelian constant group.

Given a scheme  $S$ , an  $S$ -scheme  $X$  is said to have *(tame) diagonalizable quotient singularities* if étale locally on  $S$ ,  $X$  is the quotient of a smooth  $S$ -scheme by a (tame) finite diagonalizable group scheme.

**Lemma 2.1** (Criteria for Representability). *A morphism of Artin stacks  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is representable if and only if either of the following equivalent conditions hold:*

1. *the geometric fibers of  $f$  are algebraic spaces;*
2. *for every geometric point  $x$  of  $\mathcal{X}$ , the induced map of stabilizer groups  $\text{Stab}_{\mathcal{X}}(x) \rightarrow \text{Stab}_{\mathcal{Y}}(f(x))$  is injective.*

*Proof.* The equivalence of (1) with representability of  $f$  is shown in [Con07, Corollary 2.2.7]. To show the equivalence of (1) and (2), it suffices to base change by the map  $\text{Spec } k \rightarrow \mathcal{Y}$  determined by  $f(x)$ . We can therefore assume  $\mathcal{Y}$  is the spectrum of an algebraically closed field. In this case, we must show that  $\mathcal{X}$  is an algebraic space if and only if its stabilizers at all geometric points are trivial. This is shown in [Con07, Theorem 2.2.5(1)].  $\square$

**Corollary 2.2.** *Suppose  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is a morphism of algebraic stacks.*

1. *If  $g: \mathcal{Y} \rightarrow \mathcal{Z}$  is a representable morphism of algebraic stacks, then  $f$  is representable if and only if  $g \circ f$  is representable.*
2. *If  $f$  has a section, it is representable.*
3. *If  $\mathcal{Z} \rightarrow \mathcal{Y}$  is a surjective locally finite type morphism from an algebraic stack, then  $f$  is representable if and only if  $f_{\mathcal{Z}}: \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$  is representable.*

*Proof.* (1) and (2) follow immediately from Lemma 2.1(2). (3) follows from Lemma 2.1(1), as the geometric fibers of  $f$  and  $f_{\mathcal{Z}}$  are the same.  $\square$

**Lemma 2.3.** *Suppose  $\mathcal{X}$  is an algebraic stack and  $G$  is a locally finite type group scheme over an algebraic space  $S$ . If  $U \rightarrow \mathcal{X}$  is a  $G$ -torsor, then the corresponding morphism  $\mathcal{X} \rightarrow BG$  is representable if and only if  $U$  is an algebraic space.*

*Proof.* The following diagram is cartesian.

$$\begin{array}{ccc} U & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & BG \end{array}$$

By Corollary 2.2(3),  $\mathcal{X} \rightarrow BG$  is representable if and only if  $U \rightarrow S$  is representable (i.e. if and only if  $U$  is an algebraic space).  $\square$

The following proposition is a variant of [EHKV01, Lemma 2.12].

**Proposition 2.4.** *Let  $\mathcal{X}$  be an algebraic stack. Then  $\mathcal{X} \cong [V/G]$  for some algebraic space  $V$  and subgroup  $G \subseteq GL_r$  if and only if there is a rank  $r$  vector bundle  $\mathcal{E}$  on  $\mathcal{X}$  such that the stabilizers at geometric points of  $\mathcal{X}$  act faithfully on the fibers of  $\mathcal{E}$ . Moreover,  $G$  can be taken to be  $\mathbb{G}_m^r$  exactly when  $\mathcal{E}$  can be taken to be the direct sum of line bundles; and  $G$  can be taken to be a finite diagonalizable group scheme exactly when  $\mathcal{E}$  can be taken to be the direct sum of torsion line bundles.*

*Remark 2.5.* As the proof of Proposition 2.4 shows, the vector bundle  $\mathcal{E}$  is the pullback of a universal bundle on  $BG$  along the morphism  $\mathcal{X} \rightarrow BG$  corresponding to the  $G$ -torsor  $V \rightarrow \mathcal{X}$ .

*Proof.* By Lemma 2.3, an isomorphism  $\mathcal{X} \cong [V/G]$  where  $V$  is an algebraic space is equivalent to a representable morphism  $f: \mathcal{X} \rightarrow BG$ . Given a morphism  $\mathcal{X} \rightarrow BG$ , let  $\mathcal{E}$  be the pullback of the universal rank  $r$  vector bundle on  $BGL_r$  for  $G \subseteq GL_r$  (resp. the universal sum of line bundles on  $B\mathbb{G}_m^r$  for  $G = \mathbb{G}_m^r$ , resp. the universal sum of torsion line bundles on  $BG$  for  $G \subseteq \mathbb{G}_m^r$  finite diagonalizable). For a geometric point  $x$  of  $\mathcal{X}$ , the action of  $\text{Stab}_{\mathcal{X}}(x)$  on the fiber of  $\mathcal{E}$  at  $x$  is given by the morphism of stabilizers induced by  $\mathcal{X} \rightarrow BG$ . By Lemma 2.1(2), the stabilizer action on the fibers is faithful at all geometric points if and only if  $\mathcal{X} \rightarrow BG$  is representable.  $\square$

For Bertini arguments in positive characteristic, it is important to restrict attention to the following class of morphisms.

**Definition 2.6.** A morphism  $f: X \rightarrow Y$  of schemes is *residually separable* if for all  $x \in X$ , the induced extension of residue fields  $k(x)/k(f(x))$  is separable (i.e.  $\text{Spec } k(x) \rightarrow \text{Spec } k(f(x))$  is geometrically reduced). A line bundle  $\mathcal{L}$  on  $X$  is called *residually separable* if the map to projective space induced by some finite-dimensional base-point free linear system of  $\mathcal{L}$  is residually separable.

A morphism of Deligne-Mumford stacks is *residually separable* if its geometric fibers have residually separable étale covers.

By (2) and (3) of the following lemma, the definition of residually separable morphisms of Deligne-Mumford stacks is compatible with the definition for schemes.

**Lemma 2.7.** *Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be morphisms of schemes.*

1. *If  $f$  and  $g$  are residually separable, then  $g \circ f$  is as well.*
2. *If  $f$  is an étale cover, then  $g$  is residually separable if and only if  $g \circ f$  is.*
3.  *$f$  is residually separable if and only if all of its geometric fibers are.*

*Proof.* By [StPrj, Lemma 035Z (2)], a scheme  $W$  over a field  $k$  is geometrically reduced if and only if  $W \times_k V$  is reduced for all reduced  $k$ -schemes  $V$ . From this, (1) follows easily. Since reducedness may be checked étale locally, we have (2). Finally, (3) is immediate from the definition of geometric reducedness.  $\square$

**Corollary 2.8.** *If  $\mathcal{X}$  is a tame locally finitely presented Deligne-Mumford stack with finite diagonal, then its coarse space map  $\mathcal{X} \rightarrow X$  is residually separable.*

*Proof.* Coarse space morphisms of tame Deligne-Mumford stacks are stable under arbitrary base change [Alp08, 4.7(1)], so we may assume  $X = \operatorname{Spec} k$  with  $k$  algebraically closed. Let  $U \rightarrow \mathcal{X}$  be any étale cover. Then the residue fields of  $U$  are finite over  $X$  as  $\mathcal{X} \rightarrow X$  is quasi-finite [Con, Theorem 1.1], so the residue field extensions are trivial.  $\square$

With the above preliminaries in place, we now turn to Theorem 2.9.

**Theorem 2.9.** *Let  $k$  be an infinite field and let  $X = V/H$  be an algebraic space, where  $V$  is a quasi-compact smooth algebraic space and  $H$  is a diagonalizable group scheme over  $k$  which acts properly on  $V$  with tame finite stabilizers. Further assume that every coset of  $n \cdot \operatorname{Pic}(X) \subseteq \operatorname{Pic}(X)$  contains a residually separable base-point free line bundle, where  $n$  is the least common multiple of the exponents of the stabilizers of the  $H$ -action.<sup>1</sup> Then  $X = U/G$ , where  $U$  is a smooth algebraic space and  $G$  is a finite diagonalizable group scheme over  $k$  acting properly on  $U$ .*

*Remark 2.10.* If  $X$  is quasi-projective, then every coset of  $n \cdot \operatorname{Pic}(X) \subseteq \operatorname{Pic}(X)$  automatically contains a residually separable base-point free line bundle. Indeed, every line bundle  $\mathcal{L}$  on  $X$  can be made very ample, and hence base-point free, after twisting by a sufficiently high power of  $\mathcal{O}_X(n) \in n \cdot \operatorname{Pic}(X)$ . Since the induced map from  $X$  to projective space is an immersion, the residue field extensions are trivial, so very ample line bundles are residually separable.

*Remark 2.11.* Since  $U \rightarrow [U/G]$  is a  $G$ -torsor (so affine) and  $[U/G] \rightarrow X$  is a coarse space morphism (so is cohomologically affine), we have that  $U \rightarrow X$  is affine. Thus, if  $X$  is a variety, then  $U$  is a variety as well. This shows that (2) implies (1) in Theorem 1.1.

*Proof of Theorem 2.9.* To show  $X$  is a global quotient of a smooth algebraic space by a finite diagonalizable group, by Proposition 2.4, it suffices to find a smooth Deligne-Mumford stack  $\mathcal{Y}$  with coarse space  $X$ , and torsion line bundles on  $\mathcal{Y}$  such that the stabilizers of  $\mathcal{Y}$  act faithfully on the fibers of the direct sum of the line bundles.

By hypothesis,  $X = V/H$  where  $V$  is a smooth algebraic space and  $H$  is a diagonalizable group scheme. Properness of the action of  $H$  on  $V$  is equivalent to the condition that the stack quotient  $\mathcal{X} = [V/H]$  is separated. By Proposition 2.4, there are line bundles  $\mathcal{M}_1, \dots, \mathcal{M}_r$  on  $\mathcal{X}$  so that the stabilizers of  $\mathcal{X}$  at geometric points act faithfully on the fibers of  $\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_r$ . The stabilizers act trivially on  $\mathcal{M}_i^{\otimes n}$ , so by [Alp08, Thm 10.3] there is a line bundle  $\mathcal{K}_i$  on  $X$  such that

$$\mathcal{M}_i^{\otimes n} = \phi^* \mathcal{K}_i,$$

where  $\phi : \mathcal{X} \rightarrow X$  is the coarse space map.

*Remark 2.12.* Instead of choosing a single  $n$ , we may choose  $n_i$  such that  $\mathcal{M}_i^{\otimes n_i}$  has trivial residual representations. This is desirable if one is explicitly constructing  $U$  and  $G$  for a specific  $X$ , but to make the proof more readable, we use a single  $n$ .  $\diamond$

By hypothesis, for each  $i$  there exists  $\mathcal{N}_i \in \operatorname{Pic}(X)$  such that  $\mathcal{K}_i \otimes \mathcal{N}_i^{\otimes n}$  is base-point free and residually separable. Since the stabilizers of  $\mathcal{X}$  act trivially on the fibers of  $\phi^* \mathcal{N}_i$ , the residual representations of  $\mathcal{M}_i \otimes \phi^* \mathcal{N}_i$  are isomorphic to the residual representations of  $\mathcal{M}_i$ . Replacing  $\mathcal{M}_i$

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<sup>1</sup>To see  $n$  is finite, note that  $\mathcal{X} := [V/H]$  is Deligne-Mumford with finite diagonal. Therefore, there is an étale cover  $\{X_i \rightarrow X\}_i$  such that  $\mathcal{X} \times_X X_i = [U_i/G_i]$  for an algebraic space  $U_i$  and finite group  $G_i$ . The exponents of the stabilizers of geometric points of  $\mathcal{X}_i$  divide the exponents of  $G_i$ . Since  $\mathcal{X}$  is quasi-compact, we can take the cover of  $X$  to be finite.

by  $\mathcal{M}_i \otimes \phi^* \mathcal{N}_i$ , we may therefore assume that  $\mathcal{K}_i$  is base-point free and residually separable for each  $i$ .

Let  $\Phi : W \rightarrow \mathcal{X}$  be an étale cover by a scheme and let  $\psi_i : W \rightarrow \mathbb{P}^{d_i}$  denote the composite of  $\Phi$ , the coarse space map  $\phi : \mathcal{X} \rightarrow X$ , and the map defined by a base-point free, residually separable linear system of  $\mathcal{K}_i$ . Since  $\Phi$  is étale and representable, it is residually separable. Since  $\mathcal{X}$  is a tame Deligne-Mumford stack, Corollary 2.8 shows that  $\phi$  is residually separable. It follows from Lemma 2.7(1) that  $\psi_i$  is residually separable as well.

Applying [Spr98, Corollary 4.3] to  $\psi_i$ , we see that a generic section of  $\psi_i^* \mathcal{K}_i = \Phi^*(\mathcal{M}_i)^{\otimes n}$  has smooth vanishing locus. Let  $Z \subseteq V$  be the closed locus where  $H$  does not act freely, let  $\mathcal{Z} = [Z/H]$ , and let  $Z' = \mathcal{Z} \times_{\mathcal{X}} W$ . We may then choose sections  $s_{i,1}, \dots, s_{i,c_i}$  of  $\psi_i^* \mathcal{N}_i$  for each  $i$  satisfying the following properties: the Cartier divisor  $D_{i,j}$  defined by the vanishing of  $s_{i,j}$  is smooth for each  $i$  and  $j$ , the set of divisors  $\{D_{i,j}\}_{i,j}$  have simple normal crossings, and  $Z' \cap \bigcap_j D_{i,j} = \emptyset$  for each  $i$ . Since  $D_{i,j}$  is obtained as the pullback of a hyperplane section of  $X$ , by descent, we therefore obtain smooth Cartier divisors  $\mathcal{D}_{i,j}$  on  $\mathcal{X}$  with simple normal crossings such that for each  $i$ ,  $\mathcal{Z} \cap \bigcap_j \mathcal{D}_{i,j} = \emptyset$ . Let  $\pi : \mathcal{Y} \rightarrow \mathcal{X}$  denote the  $n$ -th root construction along each of the  $\mathcal{D}_{i,j}$  (see for example [FMN10, 1.3.b]). Since the  $\mathcal{D}_{i,j}$  have simple normal crossings,  $\mathcal{Y}$  is smooth by [FMN10, 1.3.b.(3)].<sup>2</sup>

Let  $\mathcal{M}'_{i,j}$  denote the universal line bundles on  $\mathcal{Y}$  such that

$$(\mathcal{M}'_{i,j})^{\otimes n} = \pi^* \mathcal{M}_i^{\otimes n},$$

and let

$$\mathcal{L}_{i,j} := \mathcal{M}'_{i,j} \otimes \pi^* \mathcal{M}_i^{\vee}.$$

We show that the stabilizers of  $\mathcal{Y}$  act faithfully on the fibers of  $\bigoplus_{i,j} \mathcal{L}_{i,j}$ . Since the  $\mathcal{L}_{i,j}$  are torsion, Proposition 2.4 will show that  $\mathcal{Y}$  is a quotient of a smooth algebraic space by a finite diagonalizable group scheme; moreover, since  $\mathcal{Y}$  is separated, the group action is proper. To see that the stabilizer actions are faithful, we show that the corresponding map  $\mathcal{Y} \rightarrow B\mathbb{G}_m^C$  is representable, where  $C = \sum_i c_i$  is the total number of  $s_{i,j}$ . Consider the following diagram in which the square is cartesian.

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{g} & [\mathbb{A}^1/\mathbb{G}_m]^C & \xrightarrow{u} & B\mathbb{G}_m^C \\ \pi \downarrow & & \downarrow \hat{n} & & \\ B\mathbb{G}_m^r & \xleftarrow{m} & \mathcal{X} & \xrightarrow{f} & [\mathbb{A}^1/\mathbb{G}_m]^C \end{array}$$

The morphism  $m$  corresponds to the  $\mathcal{M}_i$ ,  $u \circ g$  corresponds to the  $\mathcal{M}'_{i,j}$ ,  $f$  corresponds to the  $\mathcal{D}_{i,j}$ ,  $u$  is the forgetful map which sends  $C$  line bundles with sections to the underlying line bundles, and  $\hat{n}$  sends  $C$  line bundles with sections  $(\mathcal{E}_i, t_i)$  to their  $n$ -th tensor powers  $(\mathcal{E}_i^{\otimes n}, t_i^{\otimes n})$ .

<sup>2</sup>We choose the  $\mathcal{D}_{i,j}$  to have simple normal crossings so that  $\mathcal{Y}$  is smooth, but it is possible to build  $\mathcal{Y}$  by rooting one divisor at a time, in which case smoothness of that divisor is sufficient to ensure every step is smooth. Here is a sketch of the argument. Applying [Spr98, Corollary 4.3], we may choose a section  $s_{1,1}$  of  $(\Phi \circ \phi)^* \mathcal{K}_1$  so that its vanishing divisor  $D_{1,1}$  is smooth and intersects  $Z$  properly (for a generic choice of  $s_{1,1}$ ,  $D_{1,1}$  will be smooth, and avoiding a given point on each connected component of  $Z$  is an open condition). This  $D_{1,1}$  is  $H$ -invariant, so it descends to a smooth Cartier divisor  $\mathcal{D}_{1,1}$  on  $\mathcal{X}$ . Let  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  be the  $n$ -th root stack of  $\mathcal{X}$  along  $\mathcal{D}_{1,1}$ . Since  $\mathcal{X}$  and  $\mathcal{D}_{1,1}$  are smooth,  $\mathcal{X}'$  is smooth by [FMN10, 1.3.b.(3)]. One shows that this  $\mathcal{X}'$  is a quotient of some smooth scheme  $V'$  by a torus using the same argument that is used for  $\mathcal{Y}$  in the remainder of the proof of Theorem 2.9. Now we replace  $\mathcal{X}$ ,  $V$ , and  $Z$  by  $\mathcal{X}'$ ,  $V'$ , and the preimage of  $[(Z \cap D_{1,1})/H]$  in  $V'$  and repeat the argument, producing  $D_{1,j}$ , until  $Z$  is empty. Then we replace  $Z$  by the preimage of the *original*  $Z$  and repeat the argument for  $\mathcal{K}_2$ ,  $\mathcal{K}_3$ , etc.

Proposition 2.4 shows that  $m$  is representable. Lemma 2.1(1) shows that  $u$  is representable, as the pullback of  $u$  under the universal  $\mathbb{G}_m^C$ -torsor is  $(\mathbb{A}^1)^C$ . From the following cartesian diagram and the fact that the diagonal map  $\Delta$  is representable, we see that  $(\pi, g)$  is representable.

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{(\pi, g)} & \mathcal{X} \times [\mathbb{A}^1/\mathbb{G}_m]^C \\ \downarrow & & \downarrow f \times \gamma_n \\ [\mathbb{A}^1/\mathbb{G}_m]^C & \xrightarrow{\Delta} & [\mathbb{A}^1/\mathbb{G}_m]^C \times [\mathbb{A}^1/\mathbb{G}_m]^C \end{array}$$

We see that the morphism  $(m \circ \pi, u \circ g): \mathcal{Y} \rightarrow B\mathbb{G}_m^r \times B\mathbb{G}_m^C$  corresponding to  $\bigoplus \pi^* \mathcal{M}_i \oplus \bigoplus \mathcal{M}'_{i,j}$  is representable, as it is the composite of representable morphisms  $(m, \text{id}) \circ (\text{id}, u) \circ (\pi, g)$ .

To show that  $\ell: \mathcal{Y} \rightarrow B\mathbb{G}_m^C$  corresponding to  $\bigoplus_{i,j} \mathcal{L}_{i,j}$  is representable, we may restrict to the residual gerbe of a geometric point  $y$  of  $\mathcal{Y}$  by Lemma 2.1(2). By Corollary 2.2(1), it then suffices to find a representable morphism  $h: B\mathbb{G}_m^C \rightarrow B\mathbb{G}_m^{r+C}$  such that the composite  $h \circ \ell$  agrees with  $(m \circ \pi, u \circ g)$  at  $y$  (i.e. when restricted to the residual gerbe at  $y$ ).

Let  $\tilde{\mathcal{Z}} = \pi^{-1}(\mathcal{Z})$ . Suppose  $y$  is a geometric point of  $\mathcal{Y}$  in the complement of  $\tilde{\mathcal{Z}}$ . The residual representations of the  $\mathcal{M}_i$  are trivial on the complement of  $\mathcal{Z}$ , so the residual representations of the  $\pi^* \mathcal{M}_i$  are trivial on the complement of  $\tilde{\mathcal{Z}}$ , so  $\mathcal{L}_{i,j} \cong \mathcal{M}'_{i,j}$  at  $y$ . Letting  $h: B\mathbb{G}_m^C \rightarrow B\mathbb{G}_m^{r+C}$  be the morphism sending a  $C$ -tuple of line bundles  $(\mathcal{E}_1, \dots, \mathcal{E}_C)$  to  $(\mathcal{O}, \dots, \mathcal{O}, \mathcal{E}_1, \dots, \mathcal{E}_C)$ , we see that  $h$  is representable by Corollary 2.2(2), and that  $h \circ \ell = (m \circ \pi, u \circ g)$  at  $y$ .

Now suppose that  $y$  is a geometric point of  $\tilde{\mathcal{Z}}$ . For each  $i$ , there exists some  $j_i$  such that  $y$  is not contained in  $\mathcal{D}_{i,j_i}$ . We then have that the residual representation of  $\mathcal{M}_{i,j_i}$  at  $y$  is trivial, so  $\pi^* \mathcal{M}_i \cong \mathcal{L}_{i,j_i}^\vee$  at  $y$ . Let  $h_1: B\mathbb{G}_m^C \rightarrow B\mathbb{G}_m^r \times B\mathbb{G}_m^C$  be the morphism given by sending the  $C$ -tuple  $(\mathcal{L}_{i,j})_{i,j}$  to  $((\mathcal{L}_{1,j_1}^\vee, \dots, \mathcal{L}_{r,j_r}^\vee), (\mathcal{L}_{i,j})_{i,j})$ . Let  $h_2: B\mathbb{G}_m^r \times B\mathbb{G}_m^C \rightarrow B\mathbb{G}_m^r \times B\mathbb{G}_m^C$  be given by sending  $((\mathcal{M}_1, \dots, \mathcal{M}_r), (\mathcal{L}_{i,j})_{i,j})$  to  $((\mathcal{M}_1, \dots, \mathcal{M}_r), (\mathcal{L}_{i,j} \otimes \mathcal{M}_i)_{i,j})$ . Then  $h_1$  and  $h_2$  are representable by Corollary 2.2(2), so  $h = h_2 \circ h_1$  is representable. We have that  $h \circ \ell = (m \circ \pi, u \circ g)$  at  $y$ , so  $\ell$  is representable at  $y$ .  $\square$

As a consequence, we see that Question 1 has an affirmative answer for simplicial toric varieties with tame singularities.

**Corollary 2.13.** *Every quasi-projective toric variety with tame quotient singularities over an algebraically closed field  $k$  is of the form  $U/G$ , where  $U$  is a smooth  $k$ -variety and  $G$  is a finite abelian group.*

*Proof.* Since  $X$  is a toric variety with quotient singularities, the Cox construction shows  $X = V/H$ , where  $V$  is smooth and  $H$  is a diagonalizable group scheme acting on  $V$  with finite stabilizers (see, for example, §3.1 for a review of the Cox construction), so Theorem 2.9 applies.  $\square$

### 3 Explicit construction for toric varieties

To emphasize that our proof of Theorem 2.9 is constructive, we reinterpret it in the special case when  $X$  is a toric variety. The end result is the procedure described in the following theorem. In §4, we demonstrate this procedure. We urge the interested reader to look at §4 immediately, as it clarifies the meaning of the notation.



**Theorem 3.1.** *Let  $X$  be a quasi-projective toric variety with tame quotient singularities over an infinite field  $k$ . Let  $\Sigma$  be the fan of  $X$ , let  $Z \subseteq X$  be the singular locus, and let  $X = V/H$  be the Cox construction of  $X$  (see §3.1).*

1. *There exist Weil divisors  $D_1, \dots, D_r$  which generate the class groups of all torus-invariant open affine subvarieties of  $X$ . Letting  $n_i$  be integers so that  $n_i D_i$  is Cartier for each  $i$ , the  $D_i$  can be chosen so that  $n_i D_i$  is very ample.*
2. *There exist sections  $\{s_{i,j}\}_{1 \leq j \leq c_i}$  of  $\mathcal{O}_X(n_i D_i)$  so that the preimages of the vanishing loci  $\{V(s_{i,j})\}_{i,j}$  in  $V$  are smooth and have simple normal crossings, and for each  $i$ ,  $\bigcap_j V(s_{i,j})$  is disjoint from  $Z$ .*
3. *Let  $W$  be the toric variety with fan  $\widehat{\Sigma}$ , as described in §3.3, and let  $U_{i,j} \subseteq W$  be the  $s_{i,j}$ -cut together with its  $\mu_{n_i}$ -action (see Definition 3.13). Then the scheme-theoretic intersection  $U = \bigcap_{i,j} U_{i,j}$  in  $W$  is a smooth variety with an action of  $G = \prod \mu_{n_i}^{c_i}$ , such that  $X \cong U/G$ .*

In §3.1 we review the Cox construction, which expresses every toric variety  $X$  as a quotient  $V/H$ , where  $V$  is smooth and  $H$  is a diagonalizable group scheme. We also give a non-stacky description of how to find line bundles  $\mathcal{M}_1, \dots, \mathcal{M}_r$  on  $\mathcal{X} = [V/H]$  such that the stabilizers of  $\mathcal{X}$  act faithfully on the fibers of  $\bigoplus \mathcal{M}_i$ . Finding such line bundles is the starting point in the proof of Theorem 2.9. In §3.2, we give an explicit description of  $n$ -th root constructions of  $n$ -th tensor powers of line bundles. In §3.3, we describe the fan  $\widehat{\Sigma}$  which appears in Theorem 3.1. Lastly, in §3.4 we put these results together to show how the proof of Theorem 2.9 yields the procedure described in Theorem 3.1.

### 3.1 Canonical stacks, line bundles, and divisors

#### Canonical Stacks

Suppose  $X$  is a finite type algebraic space with tame quotient singularities. Then there exists a smooth Deligne-Mumford stack  $\mathcal{X}$  with coarse space  $X$ . Moreover,  $\mathcal{X}$  may be chosen so that étale locally around any closed point  $x$  of  $X$ ,  $\mathcal{X}$  is a quotient of a smooth space  $U$  by the action of a finite group  $H$  which fixes some closed point  $u$  in  $U$  and acts faithfully away from a subspace of codimension at least 2 [Vis89, 2.9 and the proof of 2.8]. We refer to  $\mathcal{X}$  as the *canonical stack* over  $X$ .

*Remark 3.2* (Universal property of canonical stacks). If  $\mathcal{X}$  is any smooth Deligne-Mumford stack with coarse space  $X$  so that the coarse space morphism  $\pi: \mathcal{X} \rightarrow X$  is an isomorphism away from a locus of codimension at least 2, then  $\mathcal{X}$  is the canonical stack of  $X$ . Indeed, by [FMN10, Theorem 4.6], such a stack is universal (terminal) among smooth Deligne-Mumford stacks with trivial generic stabilizer and with a dominant codimension-preserving morphism to  $X$ . Note that this universal property is stable under base change by open immersions. That is, for  $U \subseteq X$  open,  $\mathcal{X} \times_X U$  is the canonical stack over  $U$ .

*Remark 3.3* (Local structure of canonical stacks). Suppose  $V$  is a vector space with an action of a finite group  $G$  with  $\text{char}(k) \nmid |G|$  so that  $\widehat{\mathcal{O}}_{X,x} = k[[V]]^G$ . Let  $P \subseteq G$  be the subgroup generated by *pseudoreflections*, subgroups which fix a subspace of codimension 1. Then the construction in [Vis89] has the property that  $\mathcal{X} \times_X \text{Spec } \widehat{\mathcal{O}}_{X,x} \cong [\text{Spec } k[[V]]^P / (G/P)]$ . Note that  $\text{Spec } k[[V]]^P = k[[W]]$  for some vector space  $W$  by the Chevalley-Shephard-Todd theorem.

*Remark 3.4* ( $\mathrm{Cl}(X) \cong \mathrm{Pic}(\mathcal{X})$ ). Let  $U \subseteq X$  be the smooth locus of  $X$ . Then  $\pi: \mathcal{X} \rightarrow X$  is an isomorphism over  $U$ . Since the complement of  $U$  in  $X$  and the complement of the preimage of  $U$  in  $\mathcal{X}$  are of codimension at least 2, we have a chain of isomorphisms

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(U) \cong \mathrm{Cl}(\mathcal{X}) \cong \mathrm{Pic}(\mathcal{X})$$

where the last isomorphism follows from the fact that every Weil divisor on  $\mathcal{X}$  is Cartier (one may check étale locally that an ideal sheaf is a line bundle, and  $\mathcal{X}$  is smooth). So, given a Weil divisor  $D$  on  $X$ , it makes sense to speak of the associated line bundle  $\mathcal{O}_{\mathcal{X}}(D)$  on  $\mathcal{X}$ .

### The Cox construction and characterization of jointly faithful line bundles

Given a normal toric variety  $X$  with no torus factors, the Cox construction [CLS11, §5.1] produces an open subscheme  $V$  of  $\mathbb{A}^n$  and a subgroup  $H \subset \mathbb{G}_m^n$  such that  $X = V/H$ . We briefly recall this construction. If  $\Sigma$  is the fan of  $X$  and  $\Sigma(1)$  denotes its set of rays, then consider the polynomial ring  $k[x_\rho \mid \rho \in \Sigma(1)]$  with one variable for each ray. This polynomial ring has a natural  $\mathrm{Cl}(X)$ -grading given by the map which sends  $x_\rho$  to the class of the divisor associated to  $\rho$ . We therefore obtain an action of the diagonalizable group  $H := D(\mathrm{Cl}(X))$  on  $\mathbb{A}^n = \mathrm{Spec} k[x_\rho]$ . The open subscheme  $V$  is the complement of closed subscheme defined by the ideal  $(\prod_{\rho \notin \sigma} x_\rho \mid \sigma \in \Sigma)$ .

If  $X$  is a normal toric variety, then there is a unique torus  $T'$  and toric variety  $X'$  without torus factors such that  $X \cong T' \times X'$ . Such an isomorphism, however, is non-canonical. Let  $X' = V'/H$  be the Cox construction for  $X'$ , and let  $V = T' \times V'$  with trivial  $H$ -action on the  $T'$  factor. We will say  $X = V/H$  is the Cox construction for  $X$ . Since the action of  $H$  is free away from a locus of codimension at least 2,  $\mathcal{X} = [V/H] = [V'/H] \times T'$  is the canonical stack over  $X$  by Remark 3.2.

**Lemma 3.5.** *Suppose  $D_1, \dots, D_r$  are Weil divisors of  $X$ . The stabilizers of  $\mathcal{X}$  act faithfully on  $\mathcal{O}_{\mathcal{X}}(D_1) \oplus \dots \oplus \mathcal{O}_{\mathcal{X}}(D_r)$  if and only if  $D_1, \dots, D_r$  generate the Weil class group of every torus-invariant open affine subvariety of  $X$ .*

*Proof.* If  $U$  is a torus-invariant open affine subvariety of  $X$ , then  $\mathcal{X} \times_X U$  is the canonical stack over  $U$ . We may therefore assume  $X$  is an affine toric variety. We may prove the lemma after removing a common torus factor from both  $X$  and  $\mathcal{X}$ , so we may further assume  $X$  has no torus factor.

When  $X$  is affine with no torus factor,  $V' = \mathrm{Spec} k[x_\rho]$  in the Cox construction described above. In this case, every stabilizer of  $\mathcal{X}$  is a subgroup of the stabilizer of the origin in  $V'$ , so we need only show that the stabilizer of the origin, namely  $H$ , acts faithfully on the fiber of  $\bigoplus \mathcal{O}_{\mathcal{X}}(D_i)$ . For a divisor  $D$ ,  $H$  acts on the fiber of  $\mathcal{O}_{\mathcal{X}}(D)$  by the character given by the image of  $D$  in  $\mathrm{Cl}(X) = D(H)$ . Therefore, the action of  $H$  on  $\bigoplus \mathcal{O}_{\mathcal{X}}(D_i)$  has a kernel (factors through a proper quotient) if and only if the images of the  $D_i$  in  $\mathrm{Cl}(X)$  are contained in a proper subgroup.  $\square$

*Remark 3.6.* Lemma 3.5 can be converted to the following global (but weaker) criterion. If  $D_1, \dots, D_r$  generate the quotient of the Weil class group of  $X$  by the Cartier class group of  $X$ , then they generate the Weil class group of every open affine torus-invariant subvariety  $U$  of  $X$ , and so the stabilizers of  $\mathcal{X}$  act faithfully on  $\bigoplus \mathcal{O}_{\mathcal{X}}(D_i)$ . To see this, note that the restriction morphism  $\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(U)$  is surjective, since taking the closure of a Weil divisor on  $U$  provides a section. This map sends Cartier divisors to the identity, since Cartier divisors on  $U$  are all trivial. We therefore get a surjection  $\mathrm{Cl}(X)/\mathrm{CaCl}(X) \rightarrow \mathrm{Cl}(U)$ .

### 3.2 An alternative description of $n$ -th roots of $n$ -th powers

Suppose  $X$  is a scheme (or stack),  $\mathcal{L}$  is a line bundle on  $X$ , and  $s$  is a global section of  $\mathcal{L}^{\otimes n}$ . Informally, we will show that the  $n$ -th root stack of  $\mathcal{L}^{\otimes n}$  along  $s$  is the quotient stack  $[U/\mu_n]$ , where  $U \hookrightarrow \mathbb{V}(\mathcal{L})$  is the closed subscheme (or closed substack) of the total space of  $\mathcal{L}$  cut out by the equation  $t^n = s$ , where the  $\mu_n$  acts on the “coordinate”  $t$  with weight 1.

More precisely,  $s$  corresponds to a morphism  $(\mathcal{L}^\vee)^{\otimes n} \rightarrow \mathcal{O}_X$ . This induces a surjection of sheaves of algebras  $\bigoplus_{k \geq 0} (\mathcal{L}^\vee)^{\otimes k} \rightarrow \bigoplus_{0 \leq k < n} (\mathcal{L}^\vee)^{\otimes k}$ , which induces a closed immersion

$$U := \operatorname{Spec}_X \bigoplus_{0 \leq k < n} (\mathcal{L}^\vee)^{\otimes k} \rightarrow \operatorname{Spec}_X \bigoplus_{k \geq 0} (\mathcal{L}^\vee)^{\otimes k} = \mathbb{V}(\mathcal{L}).$$

The  $\mathbb{Z}/n\mathbb{Z}$  grading on  $\bigoplus_{0 \leq k < n} (\mathcal{L}^\vee)^{\otimes k}$ , with 0-graded piece  $\mathcal{O}_X$ , corresponds to an action of  $\mu_n$  on  $U$  with quotient  $X$ .

**Definition 3.7.** With the notation above, we say  $U$  is the  $s$ -slice of  $\mathbb{V}(\mathcal{L})$ .

*Remark 3.8.* The section  $s: \mathcal{O} \rightarrow \mathcal{L}^{\otimes n}$  induces a surjection of sheaves of algebras  $\bigoplus_{k \geq 0} (\mathcal{L}^\vee)^{\otimes kn} \rightarrow \mathcal{O}$ , which corresponds to a closed immersion  $X \rightarrow \mathbb{V}(\mathcal{L}^{\otimes n})$ . We have the “ $n$ -th power map”  $\mathbb{V}(\mathcal{L}) \rightarrow \mathbb{V}(\mathcal{L}^{\otimes n})$ , given by the inclusion of sheaves of algebras  $\bigoplus_{k \geq 0} (\mathcal{L}^\vee)^{\otimes kn} \hookrightarrow \bigoplus_{k \geq 0} (\mathcal{L}^\vee)^{\otimes k}$ . The  $s$ -slice of  $\mathbb{V}(\mathcal{L})$  is simply the pullback of the closed immersion by the  $n$ -th power map, i.e. it is the closed subscheme  $\mathbb{V}(\mathcal{L}) \otimes_{\mathbb{V}(\mathcal{L}^{\otimes n})} X$  of  $\mathbb{V}(\mathcal{L})$ . It is clear that it is invariant under  $\mu_n \subseteq \mathbb{G}_m$ , acting as usual on  $\mathbb{V}(\mathcal{L})$ .

**Lemma 3.9.** *With the notation above, the  $n$ -th root stack of  $\mathcal{L}^{\otimes n}$  along  $s$  is the quotient stack  $[U/\mu_n]$ .*

*Proof.* First we consider the case when  $\mathcal{L} = \mathcal{O}_X$ . In this case,  $\mathbb{V}(\mathcal{L}) = \mathbb{A}_X^1$ , and  $U = V(t^n - s)$ , where  $t$  is the coordinate on  $\mathbb{A}^1$ . From the following cartesian diagram, we see that the root stack  $\sqrt[n]{(\mathcal{O}_X, s)/X}$  is precisely  $[U/\mu_n]$ .

$$\begin{array}{ccccc} U = V(t^n - s) & \longrightarrow & \mathbb{A}_t^1 & & \\ \downarrow & & \downarrow \mu_n\text{-torsor} & & \\ \sqrt[n]{(\mathcal{O}_X, s)/X} & \longrightarrow & [\mathbb{A}^1/\mu_n] & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow & & \downarrow \gamma_n \\ X & \xrightarrow{s} & \mathbb{A}^1 & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

In general, we work on the  $\mathbb{G}_m$ -torsor corresponding to  $\mathcal{L}$ , on which  $\mathcal{L}$  is canonically trivialized. Since the construction of  $[U/\mu_n]$ ,  $\sqrt[n]{(\mathcal{L}^{\otimes n}, s)/X}$ , and the morphism between them is stable under base change, the result follows from the case  $\mathcal{L} = \mathcal{O}_X$  by descent.  $\square$

*Remark 3.10.* Given this description of  $n$ -th root stacks of  $n$ -th tensor powers of line bundles, we see that the proof of Theorem 2.9 can be interpreted as an application of the philosophy in [KV04]. The basic idea of [KV04] is as follows. Start with a smooth Deligne-Mumford stack  $\mathcal{X}$  and a vector bundle  $\mathcal{E}$  on  $\mathcal{X}$  so that the stabilizers act faithfully on fibers at geometric points. The goal is to produce a smooth scheme  $U$  with a finite surjection onto  $\mathcal{X}$ . The codimension of the stacky locus

in the fibers of the total space of  $\mathcal{E}^{\oplus k}$  is large for large  $k$ , so after repeated slicing by hyperplanes, one can find a slice  $U$  of the total space which is smooth, misses the stacky locus, and is finite over  $\mathcal{X}$ .<sup>1</sup>

In our situation, we aim to produce a smooth scheme  $U$  and a finite surjection onto  $\mathcal{X}$ , which is generically a torsor under some finite group. Our strategy is to start with line bundles  $\mathcal{L}_1, \dots, \mathcal{L}_r$  so that the stacky locus of the total space of  $\bigoplus \mathcal{L}_i$  has positive codimension in each fiber (this is the condition that the stabilizers act faithfully on the fibers). Then the stacky locus of the total space of  $\bigoplus \mathcal{L}_i^{\oplus c_i}$  will have large codimension in each fiber. We then choose slices of this total space which have  $\mu_n$ -actions so that the intersection of all of the slices is smooth, misses the stacky locus, and is generically a  $\mu_n^C$ -torsor over  $\mathcal{X}$ .  $\diamond$

*Remark 3.11.* In the case when  $X$  is a stack, we can concretely describe the stabilizers of points of  $U$ . Let  $\pi : U \rightarrow X$  be the composite of the closed immersion  $U \rightarrow \mathbb{V}(\mathcal{L})$  and the projection  $\mathbb{V}(\mathcal{L}) \rightarrow X$ . If  $u$  is a geometric point of  $U$  such that  $s$  does not vanish at  $\pi(u)$ , then  $\text{Stab}_U(u)$  is the kernel of the action of  $\text{Stab}_X(\pi(u))$  on the fiber  $\mathcal{L}_{\pi(u)}$ .

To see this, first note that  $u$  maps to the complement of the zero section in  $\mathbb{V}(\mathcal{L})$ . Since  $U \rightarrow \mathbb{V}(\mathcal{L})$  is a closed immersion, and  $\mathbb{V}(\mathcal{L}) \setminus \{0\} \rightarrow \mathbb{V}(\mathcal{L})$  is an open immersion, we see  $\text{Stab}_U(u) = \text{Stab}_{\mathbb{V}(\mathcal{L}) \setminus \{0\}}(u)$ . Let  $f : X \rightarrow B\mathbb{G}_m$  be the morphism corresponding to  $\mathcal{L}$ . From the cartesian diagram

$$\begin{array}{ccc} \mathbb{V}(\mathcal{L}) \setminus \{0\} & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B\mathbb{G}_m \end{array}$$

we see that  $\text{Stab}_{\mathbb{V}(\mathcal{L}) \setminus \{0\}}(u)$  is the kernel of  $\text{Stab}_X(\pi(u)) \rightarrow \text{Stab}_{B\mathbb{G}_m}(f(\pi(u)))$ , which is precisely the kernel of the action of  $\text{Stab}_X(\pi(u))$  on the fiber  $\mathcal{L}_{\pi(u)}$ .  $\diamond$

### 3.3 Cuts of the coarse space of $\mathbb{V}(\bigoplus \mathcal{O}_{\mathcal{X}}(D_i))$

Let  $D_1, \dots, D_r$  be (not necessarily distinct) divisors on  $X$ . For each  $i$ , let  $D_i = \sum c_{i,\rho} D_\rho$ , where  $D_\rho$  is the divisor associated to the ray  $\rho$  of  $\Sigma$ . Given a ray  $\rho$  of  $\Sigma$ , let  $\lambda_\rho \in N$  be the first lattice point along  $\rho$ , and define  $\hat{\rho}$  to be the ray in  $\mathbb{Z}^r \oplus N$  spanned by  $e_i - \sum c_{i,\rho} \lambda_\rho$ . Given a cone  $\sigma$  of  $\Sigma$ , define  $\hat{\sigma}$  to be the cone generated by  $\{e_1, \dots, e_r\}$  and  $\{\hat{\rho} | \rho \text{ a ray of } \sigma\}$ . Let  $\hat{\Sigma}$  be the fan on  $\mathbb{Z}^r \oplus N$  generated by the cones  $\{\hat{\sigma} | \sigma \in \Sigma\}$ . Let  $W$  be the toric variety with fan  $\hat{\Sigma}$ , and let  $D'_i$  be the divisor in  $W$  corresponding to the ray  $e_i$ . Let  $p : W \rightarrow X$  be the morphism induced by the projection  $\hat{\Sigma} \rightarrow \Sigma$ .

This generalizes the usual toric construction of the total space of the sum of a collection of line bundles. Informally,  $W$  can be thought of as the total space of the sum of a collection of Weil divisors. This is made precise by the following proposition.

**Proposition 3.12.** *Let  $\mathcal{L}_i = \mathcal{O}_{\mathcal{X}}(D_i)$ , as defined in Remark 3.4. Then  $W$  is the coarse space of  $\mathbb{V}(\bigoplus \mathcal{L}_i)$ .*

*Proof.* Let  $\pi : V \rightarrow \mathcal{X}$  be the  $H$ -torsor from the Cox construction. By descent, the data of the line bundle  $\mathcal{L}_i$  is equivalent to the data of the line bundle  $\pi^* \mathcal{L}_i$  with an  $H$ -linearization, since  $V$

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<sup>1</sup>Instead of the total space of  $\mathcal{E}^{\oplus k}$ , [KV04] uses a large fiber product of the projective bundle of  $\mathcal{E}$ . This is necessary to ensure that enough sections exist. In our situation, we twist by a large power of a very ample line bundle instead.

is an open subset of  $\mathbb{A}^{\Sigma(1)}$ , all line bundles on it are trivial, and the weight of the  $H$ -action on the coordinate corresponding to  $\mathcal{L}_i$  is  $D_i \in \text{Cl}(X) = D(H)$ . Letting  $\widehat{V} = \mathbb{V}(\bigoplus \pi^* \mathcal{L}_i)$ , we have that  $\mathbb{V}(\bigoplus \mathcal{L}_i) = [\widehat{V}/H]$ . We will show that this quotient structure naturally agrees with the Cox construction of  $W$ .

Every maximal cone of  $\widehat{\Sigma}$  contains all the  $e_i$ , so the open subset of  $\mathbb{A}^{\widehat{\Sigma}(1)}$  in the Cox construction of  $W$  is determined by sets of rays  $\{\hat{\rho}_1, \dots, \hat{\rho}_m\}$  which do not lie on a single cone of  $\widehat{\Sigma}$ . Such a subset of rays fails to lie on a single cone of  $\widehat{\Sigma}$  if and only if the corresponding subset  $\{\rho_1, \dots, \rho_m\}$  fail to lie on a single cone of  $\Sigma$ . This shows that  $\widehat{V}$  is the same open subset of  $\mathbb{A}^{\widehat{\Sigma}(1)}$  as in the Cox construction of  $W$ . It is straightforward to see that the  $H$ -actions agree.  $\square$

Now suppose  $n_i D_i$  is a very ample Cartier divisor on  $X$ , with corresponding lattice polyhedron  $P_i = \{x \in N^\vee \otimes \mathbb{R} \mid \langle x, \rho \rangle \geq -n_i c_{i,\rho}\}$ . Consider the polyhedron

$$P'_i = \{x \in (\mathbb{Z}^r \oplus N)^\vee \otimes \mathbb{R} \mid \langle x, \rho \rangle \geq -n_i c_{i,\rho}, \langle x, e_j \rangle = 0 \text{ for } j \neq i, \text{ and } \langle x, e_i \rangle \geq 0\}.$$

The lattice points in  $P'_i$  correspond to torus semi-invariant sections of  $p^* \mathcal{O}_X(n_i D_i)$  which do not vanish along any  $D'_j$  with  $j \neq i$ . We make two observations:

1. There is a natural identification of  $P_i$  with  $\{x \in P'_i \mid \langle x, e_i \rangle = 0\}$ . Geometrically, pullback induces an isomorphism between the sections of  $\mathcal{O}_X(n_i D_i)$  and the sections of  $\mathcal{O}_W(n_i D_i)$  which are linear combinations of torus semi-invariant sections not vanishing along any of the  $D'_j$ .
2. There is a point  $x \in P'_i$  with  $\langle x, e_i \rangle = n_i$  and  $\langle x, \rho \rangle = 0$ . This follows immediately from the fact that  $n_i D'_i$  is linearly equivalent to  $n_i D_i$ . Geometrically, there is a section of  $\mathcal{O}_W(n_i D_i)$  which induces the divisor  $n_i D'_i$ . We will denote this section by  $t_i$ .

Concretely,  $P'_i$  is a pyramid of height  $n_i$  with base  $P_i$ . The apex of the pyramid is the lattice point corresponding to  $t_i$ . If  $s$  is a linear combination of torus semi-invariant sections corresponding to the lattice points in  $P_i$ , then  $p^*(s)$  is “the same” linear combination of the torus semi-invariant sections corresponding to the lattice points in the base of  $P'_i$ .

**Definition 3.13.** Suppose  $s$  is a section of  $\mathcal{O}_X(n_i D_i)$ . The  $s$ -cut of  $W$  is the vanishing locus of the section  $t_i - p^*(s)$  of  $\mathcal{O}_W(n_i D_i)$ .

*Remark 3.14* ( $\mu_{n_i}$ -action on the cut). The 1-parameter subgroup corresponding to  $e_i$  acts with weight 0 on  $p^*(s)$  and with weight  $n_i$  on  $t_i$ , so the  $s$ -cut of  $W$  is invariant under the action of  $\mu_{n_i}$ .

*Remark 3.15* (compatibility of terminology). The pyramid of height 1 and base  $P_i$  corresponds to (a compactification of) the total space of the line bundle  $\mathcal{O}_X(n_i D_i)$ . For a section  $s$  of  $\mathcal{O}_X(n_i D_i)$ , the corresponding closed subscheme  $X \hookrightarrow \mathbb{V}(\mathcal{O}_X(n_i D_i))$  is given by the difference of the torus semi-invariant section corresponding to the apex and the pullback of  $s$ . The  $n_i$ -th power map  $\mathbb{V}(\mathcal{O}_X(D_i)) \rightarrow \mathbb{V}(\mathcal{O}_X(n_i D_i))$  is induced by vertically scaling the pyramid by a factor of  $\frac{1}{n_i}$ . Combining Proposition 3.12 with the description in Remark 3.8, we see that the  $s$ -cut of  $W$  is the coarse space of  $U \times \mathbb{V}(\bigoplus_{j \neq i} \mathcal{L}_j)$ , where  $U$  is the  $s$ -slice of  $\mathbb{V}(\mathcal{L}_i)$ .

### 3.4 Proof of Theorem 3.1

We now prove the explicit procedure described for toric varieties. We follow the proof of Theorem 2.9, taking into account the results in §§3.1–3.3. Let  $X = V/H$  be the Cox construction of  $X$ , and  $\mathcal{X} = [V/H]$  denote the canonical stack of  $X$ . Let  $\phi: \mathcal{X} \rightarrow X$  be the coarse space map, which is an isomorphism away from the singular locus  $Z \subseteq X$ .

(1) The irreducible torus-invariant divisors of  $X$  are  $\mathbb{Q}$ -Cartier and clearly generate the class groups of all torus-invariant open affine subvarieties. The  $n_i D_i$  can be assumed to be very ample since one can add an arbitrary ample divisor to each  $D_i$  without changing the fact that the  $D_i$  generate the class groups of all torus-invariant open subvarieties. By Lemma 3.5, the stabilizers of  $\mathcal{X}$  act faithfully on the fibers of  $\bigoplus \mathcal{O}_{\mathcal{X}}(D_i)$ .

(2) The  $s_{i,j}$  may be chosen to have the given properties by the same Bertini argument used in the proof of Theorem 2.9.

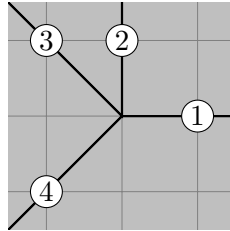
(3) Let  $\mathcal{U}_{i,j} \subseteq \mathbb{V}(\mathcal{O}_{\mathcal{X}}(D_i))$  be the  $s_{i,j}$ -slice together with the action of  $\mu_{n_i}$  (Definition 3.7). Let  $U \subseteq \mathbb{V}(\bigoplus_i \mathcal{O}_{\mathcal{X}}(D_i)^{\oplus c_i})$  be the fiber product of the  $\mathcal{U}_{i,j}$  over  $\mathcal{X}$ , and let  $\pi: U \rightarrow \mathcal{X}$  be the projection. The stacky locus of  $\mathbb{V}(\bigoplus_i \mathcal{O}_{\mathcal{X}}(D_i)^{\oplus c_i})$  lies over  $Z$ . Combining Remark 3.11 with the fact that  $\bigcap_j V(s_{i,j})$  is disjoint from  $Z$ , we see that for any geometric point  $u$  of  $U$ ,  $\text{Stab}_U(u)$  must be contained in the kernel of the action of  $\text{Stab}_{\mathcal{X}}(\pi(u))$  on  $(\bigoplus \mathcal{O}_{\mathcal{X}}(D_i))_{\pi(u)}$ . Since this action is faithful, the stabilizers of geometric points of  $U$  are trivial, so  $U$  is an algebraic space.

By Lemma 3.9, the stack  $\mathcal{U}$ , given by taking the  $n_i$ -th root of  $s_{i,j}$  for each  $i$  and  $j$ , is isomorphic to  $[U/\prod_i \mu_{n_i}^{c_i}]$ . By Proposition 3.12, the coarse space of  $\mathbb{V}(\bigoplus_i \mathcal{O}_{\mathcal{X}}(D_i)^{\oplus c_i})$  is  $W$ , and by Remark 3.15, the intersection of the  $s_{i,j}$ -cuts of  $W$  (Definition 3.13) is  $U$ .

Since the preimages of  $V(s_{i,j})$  in  $\mathcal{X}$  are smooth and have simple normal crossings, the root stack  $\mathcal{U}$  is smooth, so  $U$  is smooth. We therefore have that  $U$  is a smooth scheme with an action of  $\prod \mu_{n_i}^{c_i}$  such that  $X \cong U/\prod \mu_{n_i}^{c_i}$ .

## 4 Example: blow-up of $\mathbb{P}(1, 1, 2)$

In this section, we run through the procedure described in Theorem 3.1 when  $X$  is weighted projective space  $\mathbb{P}(1, 1, 2)$  blown-up at a smooth torus-invariant point. The fan of  $X$  is illustrated below.



As mentioned in the introduction, the interest in this example (among others) is that the map  $U \rightarrow X$  of Corollary 2.13 is not toric:

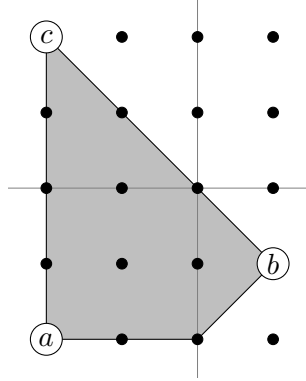
**Proposition 4.1.** *There is no toric variety  $U$  and finite subgroup  $G$  of the torus of  $U$  such that  $X = U/G$ .*

*Proof.* We show that there is no finite toric morphism  $f: U \rightarrow X$  with  $U$  a smooth toric variety. Let  $\Sigma$  (resp.  $\Sigma'$ ) denote the fan of  $X$  (resp.  $U$ ) with ambient lattice  $N$  (resp.  $N'$ ). Then  $\Sigma'$  has four rays and each ray maps to a ray of  $\Sigma$ . Let  $\sigma \in \Sigma'$  be the cone which maps to the cone of  $\Sigma$

generated by  $(1, 0)$  and  $(0, 1)$ . Let  $v_1$  and  $v_2$  be the first lattice points on the rays of  $\sigma$ . Since  $\sigma$  is smooth,  $v_1$  and  $v_2$  form a basis for  $N'$ . The map  $g : N' \rightarrow N$  induced by  $f$  is then determined by positive integers  $a$  and  $b$  such that  $g(v_1) = (a, 0)$  and  $g(v_2) = (0, b)$ . Let  $\tau \in \Sigma'$  be the cone mapping to the cone of  $\Sigma$  generated by  $(-1, 1)$  and  $(-1, -1)$ . We claim that  $\tau$  is singular. Indeed, the first lattice points on the rays of  $\tau$  are  $(bv_1 + av_2)/\gcd(a, b)$  and  $(bv_1 - av_2)/\gcd(a, b)$ , which do not form a basis for  $N'$ .  $\square$

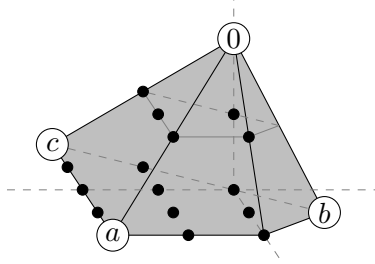
We now apply Theorem 3.1 to  $X$ .

**(1: choose  $D$ )** Let  $D_{\rho_i}$  be the divisor corresponding to the  $i$ -th ray. Then the divisor  $D = D_{\rho_1} + D_{\rho_2} + D_{\rho_3}$  generates the class group of each torus-invariant open affine. We have that  $2D$  is a very ample Cartier divisor. The lattice polytope of  $2D$  is shown below.



**(2: choose  $s$ )** The singular locus of  $X$  is a single point, the intersection of  $D_{\rho_3}$  and  $D_{\rho_4}$ . Let  $s_a$ ,  $s_b$ , and  $s_c$  be the torus semi-invariant sections of  $\mathcal{O}(2D)$  corresponding to the labeled lattice points. Pulling back to  $V \subseteq \mathbb{A}^4 = \text{Spec } k[x_1, x_2, x_3, x_4]$  in the Cox construction, we see that  $s_a$ ,  $s_b$ , and  $s_c$  pull back to  $x_3^2 x_4^4$ ,  $x_1^3 x_2$ , and  $x_2^4 x_3^6$ , respectively. It is straightforward to verify with the Jacobian criterion that the vanishing locus of  $x_3^2 x_4^4 + x_1^3 x_2 + x_2^4 x_3^6$  is smooth on  $V$  and misses the locus  $\{x_3 = x_4 = 0\}$ , so the vanishing locus of  $s = s_a + s_b + s_c$  is smooth on  $X$  and misses the singular point.

**(3: compute  $U$ )** Consider the projective toric variety in  $\mathbb{P}^{18}$  defined by the following polytope.



Note that the toric variety  $W$  defined by  $\hat{\Sigma}$  is an open subvariety, and this projective variety is precisely the closed image of the morphism  $\psi : W \rightarrow \mathbb{P}^{18}$  given by  $2D'$ . Let  $U$  be the hyperplane slice of this variety (or equivalently, of  $W$ ) defined by  $x_0 - x_a - x_b - x_c$ .

There is a  $\mu_2$ -action on  $\mathbb{P}^{18}$  given as follows: if  $\chi$  is the non-trivial character of  $\mu_2$  and  $x$  is a lattice point of the above polytope with height  $h$ , then  $\mu_2$  acts the coordinate corresponding to  $x$  through the character  $\chi^h$ . Then  $U$  is invariant under the  $\mu_2$ -action and  $X = U/\mu_2$ .

## 5 Characterization of torus quotients: proof of main theorem

Throughout this section, we work over a field  $k$ . Our goal is to prove Theorem 5.1. In §5.1, we use Theorem 5.1 to prove Theorem 1.5, thereby answering Question 2.

**Notation.** Throughout this section, we will use the notation  $\widehat{\mathcal{O}}_{X,x}$  to denote the completion of the étale local ring at a point  $x$  of an algebraic space  $X$ .<sup>12</sup>

We use the term *torus* to mean split torus. We say that a stack  $\mathcal{X}$  is a *torus quotient stack* if there is a smooth algebraic space  $V$  and a torus  $T$  acting properly on  $V$  with finite stabilizers such that  $\mathcal{X} = [V/T]$ . We say an algebraic space  $X$  is a *torus quotient space* if it is the coarse space of a torus quotient stack.

**Theorem 5.1.** *Suppose an algebraic space  $X$  has tame diagonalizable quotient singularities, and consider the following:*

1.  $X$  is a quotient of a smooth algebraic space by a proper action of a finite diagonalizable group.
2.  $X$  is a torus quotient space.
3.  $X$  has Weil divisors  $D_1, \dots, D_r$  whose images generate  $\text{Cl}(\widehat{\mathcal{O}}_{X,x})$  for all points  $x$  of  $X$ .
4. the canonical stack  $\mathcal{X}$  over  $X$  is a torus quotient stack.

Conditions (2)–(4) are equivalent and implied by (1). If  $X$  is quasi-projective and  $k$  is infinite, then (1) is equivalent to (2)–(4).

*Remark 5.2.* The image of  $\text{Cl}(X)$  in  $\text{Cl}(\widehat{\mathcal{O}}_{X,x})$  is determined by any open neighborhood of  $x$ , so (3) in the above theorem is a Zariski local condition.

*Proof of Theorem 1.1 from Theorem 5.1.* Suppose  $X$  is a quasi-projective variety and  $k$  is algebraically closed. Then tame finite diagonalizable groups are the same as tame finite abelian groups.

An action of a finite group on a separated scheme is always proper. By [Sum74, Corollary 2], for any action of a torus on a quasi-projective variety, every point has an invariant affine open neighborhood. A linearly reductive group acting on an affine scheme with tame finite stabilizers must act properly. This shows that any torus action on a quasi-projective variety with tame finite stabilizers is proper. Therefore, for  $i = 1, 2, 4$ , condition (i) of Theorem 1.1 implies condition (i) of Theorem 5.1.

If  $V$  is a smooth algebraic space with a proper action of an affine group  $H$  with tame finite stabilizers so that  $X = V/H$ , then  $V \rightarrow [V/H]$  is affine and  $[V/H] \rightarrow X$  is a coarse space morphism (so cohomologically affine), so  $V \rightarrow X$  is affine, so  $V$  is a quasi-projective variety. Therefore, for  $i = 1, 2, 4$ , condition (i) of Theorem 5.1 implies condition (i) of Theorem 1.1.

Condition (3) is the same in the two theorems. □

<sup>1</sup>The étale local ring at  $x$  is simply the strict henselisation of the local ring at a preimage of  $x$  in some étale cover of  $X$ . See [StPrj, Definition 04KG].

<sup>2</sup> If one wishes to consider only the case when  $X$  is a scheme, one may let  $\widehat{\mathcal{O}}_{X,x}$  denote the completion of the Zariski local ring in this section. The proofs apply verbatim. The étale local ring is used simply because algebraic spaces do not have Zariski local rings.



**Lemma 5.3.** *Let  $G$  be a finite diagonalizable group acting on a regular complete noetherian local  $k$ -algebra  $A$  with maximal ideal  $\mathfrak{m}$  and residue field  $K$ . There is a  $G$ -equivariant isomorphism  $(\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2))^\wedge \rightarrow A$ , where  $\mathfrak{m}/\mathfrak{m}^2$  is given the induced linear  $G$ -action. Moreover,  $A^G$  is isomorphic to the complete local ring at the distinguished point of a pointed affine toric variety over  $K$ .*

*Proof.* Since  $G$  is linearly reductive (even if it is not tame), the surjection  $\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^2$  admits a  $G$ -equivariant splitting  $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}$ . The induced ring homomorphism  $(\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2))^\wedge \rightarrow A$  is  $G$ -equivariant by construction, and induces an isomorphism of tangent spaces, so by the Cohen Structure Theorem, it is an isomorphism.

Since  $G$  is linearly reductive, the completion of  $\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2)^G$  is isomorphic to  $((\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2))^\wedge)^G$ . Thus, for the final assertion, it suffices to show that  $\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2)^G$  is the coordinate ring of a pointed affine toric variety. Since  $G$  is diagonalizable, there exists a basis  $x_1, \dots, x_n$  for  $\mathfrak{m}/\mathfrak{m}^2$  so that each  $x_i$  is semi-invariant. Let  $M$  be the monoid of monomials in the  $x_i$  which are invariant under  $G$ . It is clear that  $M$  is saturated and sharp, and that  $\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2)^G = K[x_1, \dots, x_n]^G = K[M]$ .  $\square$

**Lemma 5.4.** *Let  $U$  be a smooth algebraic space and  $G$  a diagonalizable group scheme which acts properly on  $U$  with tame finite stabilizers. Let  $\mathcal{X}$  be the canonical stack over  $U/G$  and  $f: [U/G] \rightarrow \mathcal{X}$  the induced map (see Remark 3.2). Then the ramification divisor  $\mathcal{D} \subseteq \mathcal{X}$  of  $f$  is a simple normal crossing divisor.*

*Proof.* Let  $X = U/G$  be the coarse space of  $[U/G]$ . To show that  $\mathcal{D}$  has simple normal crossings, it suffices to check that for every point  $x$  of  $X$ , the pullback of  $\mathcal{D}$  to  $\mathcal{X} \times_X \mathrm{Spec} \hat{\mathcal{O}}_{X,x}$  is a simple normal crossing divisor. We may therefore replace  $X$  by  $\mathrm{Spec} \hat{\mathcal{O}}_{X,x}$ ,  $U$  by  $\mathrm{Spec} \hat{\mathcal{O}}_{U,u}$ , and  $G$  by the stabilizer of  $u$ , where  $u$  maps to  $x$ . We may quotient by the kernel of the action of  $G$  without changing the ramification divisor  $\mathcal{D}$ , so we may assume  $G$  acts faithfully on  $U$ .

Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}_{U,u}$  and let  $K$  be the residue field of  $u$ . By Lemma 5.3,  $\mathcal{O}_{U,u}$  is  $H$ -equivariantly isomorphic to  $(\mathrm{Sym}_K^*(\mathfrak{m}/\mathfrak{m}^2))^\wedge$ . Since  $G$  is diagonalizable, there is a basis  $x_1, \dots, x_n$  for  $\mathfrak{m}/\mathfrak{m}^2$  so that each  $x_i$  is semi-invariant. Let  $P_i \subseteq G$  be the subgroup which acts trivially on  $x_j$  for all  $j \neq i$ . The  $P_i$  are precisely the pseudoreflections of the action of  $G$ . The subgroup  $P \subseteq G$  generated by pseudoreflections is the direct sum of the  $P_i$ . By Remark 3.3, we have

$$\mathcal{X} = [\mathrm{Spec} \mathcal{O}_{U,u}^P / (G/P)].$$

Since  $P$  and  $G/P$  are étale, and  $\mathcal{D}$  is the ramification divisor of

$$[U/G] = [\mathrm{Spec} \mathcal{O}_{U,u}/G] = [[\mathrm{Spec} \mathcal{O}_{U,u}/P] / (G/P)] \rightarrow [\mathrm{Spec} \mathcal{O}_{U,u}^P / (G/P)] = \mathcal{X},$$

it suffices to check that the ramification divisor  $D \subseteq \mathrm{Spec} \mathcal{O}_{U,u}^P$  of  $\pi: \mathrm{Spec} \mathcal{O}_{U,u} \rightarrow \mathrm{Spec} \mathcal{O}_{U,u}^P$  has simple normal crossings. If  $P_i$  has order  $\ell_i$ , then we see that  $\pi$  is given by the inclusion of rings  $K[[y_1, \dots, y_n]] \subseteq K[[x_1, \dots, x_n]]$  with  $y_i = x_i^{\ell_i}$ . Hence,  $D$  is the union of the  $y_i$ -coordinate hyperplanes with  $\ell_i \neq 1$ , so is a normal crossing divisor.

Before proving smoothness of the components of  $\mathcal{D}$ , we observe that the ramification divisor on the source of  $\pi$  is  $\pi^{-1}(D) \subseteq \mathrm{Spec} \mathcal{O}_{U,u}$ , given by the  $x_i$ -coordinate hyperplanes with  $\ell_i \neq 1$ . Note that the stabilizers of the components are precisely the  $P_i$ , and that these subgroups of  $G$  are distinct for distinct components of the ramification divisor.

We now show that the components of  $\mathcal{D}$  are smooth. For this, it no longer suffices to replace  $X$  by  $\mathrm{Spec} \hat{\mathcal{O}}_{X,x}$ , and so we return to the original notation. To complete the proof, we show that

for every point  $z$  of  $\mathcal{X}$ , the number of components of  $\mathcal{D}$  passing through  $z$  is equal to the number of components of the induced divisor in  $\mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ , where  $x$  is the image of  $z$  in  $X$ .

Let  $u$  be a point of  $[U/G]$  such that  $f(u) = z$ . We have that  $f$  is a homeomorphism, as  $[U/G] \rightarrow X$  and  $\mathcal{X} \rightarrow X$  are coarse space maps, hence homeomorphisms. Therefore, the components of  $\mathcal{D}$  passing through  $z$  are in bijection with the components of  $f^{-1}(\mathcal{D})$  passing through  $u$ . These are in bijection with the components of the induced divisor in  $[U/G] \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ , as the distinct formal components are stabilized by distinct subgroups of  $G$ . Finally, these are in bijection with the components of the induced divisor in  $\mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ , as the morphism  $[U/G] \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \rightarrow \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$  is a homeomorphism.  $\square$

**Corollary 5.5.** *Suppose  $U$  is a smooth algebraic space and  $G$  is a diagonalizable group that acts properly on  $U$  with tame finite stabilizers. If  $\mathcal{X}$  is the canonical stack over  $U/G$ , then the induced map  $f : [U/G] \rightarrow \mathcal{X}$  is a root stack morphism along a collection of smooth connected divisors with normal crossings.*

*Proof.* This is a direct consequence of Lemma 5.4 and Theorem 6.1, as a root stack of a simple normal crossing divisor is smooth [FMN10, 1.3.b(3)], so isomorphic to its canonical stack.  $\square$

**Corollary 5.6.** *Let  $X$  be an algebraic space with quotient singularities and let  $\mathcal{X}$  be its canonical stack. Then  $X$  is a torus quotient space if and only if  $\mathcal{X}$  is a torus quotient stack.*

*Proof.* By Corollary 5.5, it suffices to show that a stack  $\mathcal{X}$  is a torus quotient stack if and only if  $\mathcal{Y} = \sqrt[n]{(\mathcal{L}, s)}/\mathcal{X}$  is a torus quotient stack for some  $\mathcal{L}$ ,  $s$ , and  $n$ , where the vanishing locus of  $s$  is smooth and connected. Consider the following diagram, in which  $f$  corresponds to the line bundle  $\mathcal{L}$  with section  $s$ . The squares are cartesian. Let  $\mathcal{M}$  be the universal line bundle on  $\mathcal{Y}$  so that  $\mathcal{M}^{\otimes n} \cong \mathcal{L}$  (i.e. the line bundle corresponding to  $u \circ g$  in the diagram below).

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{g} & [\mathbb{A}^1/\mathbb{G}_m] & \xrightarrow{u} & B\mathbb{G}_m \\ \pi \downarrow & & \downarrow \gamma_n & & \downarrow \gamma_n \\ \mathcal{X} & \xrightarrow{f} & [\mathbb{A}^1/\mathbb{G}_m] & \xrightarrow{u} & B\mathbb{G}_m \end{array}$$

( $\Rightarrow$ ) If  $\mathcal{X}$  is a torus quotient stack, by Lemma 2.3 there exists a representable morphism  $\kappa : \mathcal{X} \rightarrow B\mathbb{G}_m^r$ . Then  $(\kappa, u \circ f) : \mathcal{X} \rightarrow B\mathbb{G}_m^{r+1}$  is also representable. Since  $(\kappa \circ \pi, u \circ g) : \mathcal{Y} \rightarrow B\mathbb{G}_m^{r+1}$  is the pullback of  $(\kappa, u \circ f)$ , it is representable, so  $\mathcal{Y}$  is a torus quotient stack by Lemma 2.3.

( $\Leftarrow$ ) Now suppose  $\mathcal{Y}$  is a torus quotient stack. Then there is a representable morphism  $\ell : \mathcal{Y} \rightarrow B\mathbb{G}_m^r$ , corresponding to a tuple of line bundles  $(\mathcal{L}_1, \dots, \mathcal{L}_r)$ . Then  $(\ell, u \circ g)$ , corresponding to  $(\mathcal{L}_1, \dots, \mathcal{L}_r, \mathcal{M})$  is representable. By [Cad07, Corollary 3.1.2], for each  $j$ , we have

$$\mathcal{L}_j = \mathcal{M}^{i_j} \otimes \pi^* \mathcal{K}_j$$

for some  $0 \leq i_j < n$  and line bundle  $\mathcal{K}_j$  on  $\mathcal{X}$ . Composing with the automorphism of  $B\mathbb{G}_m^{r+1}$  sending  $(\mathcal{L}_1, \dots, \mathcal{L}_r, \mathcal{M})$  to  $(\mathcal{L}_1 \otimes \mathcal{M}^{-i_1}, \dots, \mathcal{L}_r \otimes \mathcal{M}^{-i_r}, \mathcal{M})$ , we see that there is a representable morphism  $\mathcal{Y} \rightarrow B\mathbb{G}_m^{r+1}$  of the form  $(\kappa \circ \pi, u \circ g) : \mathcal{Y} \rightarrow B\mathbb{G}_m^{r+1}$ . This morphism is the pullback of  $(\kappa, u \circ f) : \mathcal{X} \rightarrow B\mathbb{G}_m^{r+1}$ , which must also be representable by Corollary 2.2(3). Thus,  $\mathcal{X}$  is a torus quotient stack.  $\square$

**Proposition 5.7.** *Let  $A$  be a regular ring and let  $G$  be a tame finite diagonalizable group which acts faithfully on  $X = \operatorname{Spec} A$  and freely on an open subscheme  $U \subseteq X$  whose complement has codimension at least 2. Suppose that one of the following holds:*

1.  $A$  is local with maximal ideal  $\mathfrak{m}$ , or
2.  $A = k[x_1, \dots, x_n]$  and  $G$  fixes the origin in  $\mathbb{A}^n = \operatorname{Spec} A$ .

*Then  $\operatorname{Cl}(A^G)$  is canonically isomorphic to  $D(G)$ .*

*Proof.* By assumption, the complement of  $U$  in  $X$  is of codimension at least 2, so the canonical stack over  $\operatorname{Spec} A^G$  is  $[X/G]$  by Remark 3.2. By Remark 3.4, we have  $\operatorname{Cl}(A^G) \cong \operatorname{Pic}([X/G]) \cong \operatorname{Pic}^G(X)$ . We will show that  $\operatorname{Pic}^G(X)$  is canonically isomorphic to  $D(G)$ .

We have that  $\operatorname{Pic}(X) = 0$ , so we wish to find all linearizations of  $\mathcal{O}_X$ . Twisting the trivial linearization by an element of  $D(G)$  gives us a morphism  $\iota: D(G) \rightarrow \operatorname{Pic}^G(X)$ . Given a linearization, we recover a character of  $G$  by considering the action on the fiber over the closed point of  $X$  (resp. the origin), so  $\iota$  is injective.

Now we show  $\iota$  is surjective. Let  $M$  be a free  $A$ -module of rank 1. A  $G$ -linearization of  $M$  is equivalent to a  $D(G)$ -grading  $M = \bigoplus M_\chi$  compatible with the  $D(G)$ -grading on  $A$  given by the  $G$ -action. To prove that this  $G$ -linearization is in the image of  $\iota$ , it suffices to find a semi-invariant generator of  $M$ .

Let  $m \in M$  be a generator, and let  $m = \sum m_\chi$  with  $m_\chi \in M_\chi$ . In the case where  $A$  is local with maximal ideal  $\mathfrak{m}$ , not every  $m_\chi$  can be in  $\mathfrak{m} \cdot m$ , so some  $m_\chi$  is a unit multiple of  $m$ . This  $m_\chi$  is then a semi-invariant generator of  $M$ .

In the case where  $A$  is a polynomial ring, we claim that  $m$  is already semi-invariant. Using the functor of points description of the  $G$ -action on  $A$ , this amounts to showing that for every  $k$ -algebra  $R$ , every element  $g \in G(R)$  sends  $m \otimes 1 \in M \otimes_A R[x_i]$  to  $m \otimes u$  with  $u \in R^\times$ . Since  $m \otimes 1$  generates  $M \otimes_A R[x_i]$  as an  $R[x_i]$ -module,  $g(m \otimes 1)$  is also a generator, and is therefore equal to  $u(m \otimes 1)$  with  $u \in R[x_i]^\times = R^\times$ , as desired.  $\square$

**Corollary 5.8.** *Suppose  $X$  a pointed affine toric variety with quotient singularities and distinguished point  $x$ . Then the restriction morphism  $\operatorname{Cl}(X) \rightarrow \operatorname{Cl}(\widehat{\mathcal{O}}_{X,x})$  is an isomorphism.*

*Proof.* Let  $X = V/H$  be the Cox construction of  $X$ . By Proposition 5.7,  $\operatorname{Cl}(X)$  and  $\operatorname{Cl}(\widehat{\mathcal{O}}_{X,x})$  are naturally isomorphic to  $D(H)$ , where a divisor  $D$  corresponds to the character of  $H$  given by the central fiber of  $\mathcal{O}_X(D)$ . Therefore, we see that the map  $\operatorname{Cl}(X) \rightarrow \operatorname{Cl}(\widehat{\mathcal{O}}_{X,x})$  is an isomorphism.  $\square$

*Proof of Theorem 5.1.* To see that (1) implies (2), note that if  $X = U/G$  with  $U$  smooth and  $G \subseteq \mathbb{G}_m^r$  a finite diagonalizable group scheme, then  $X = (U \times^G \mathbb{G}_m^r)/\mathbb{G}_m^r$ . Conversely, if  $X$  is quasi-projective and  $k$  is infinite, then Theorem 2.9 shows that (2) implies (1).

Corollary 5.6 shows that (2) and (4) are equivalent.

To complete the proof, we show that (3) and (4) are equivalent. By Remark 3.4 there is an equivalence between Weil divisors on  $X$  and line bundles on  $\mathcal{X}$ . By Proposition 2.4,  $\mathcal{X}$  is a torus quotient stack if and only if there is a collection of Weil divisors  $D_1, \dots, D_r$  on  $X$  such that the residual representations of  $\bigoplus \mathcal{O}_X(D_i)$  are faithful. To check faithfulness at a geometric point  $z$  of  $\mathcal{X}$ , it suffices to do so after base changing by  $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \rightarrow X$ , where  $x \in X$  is the image of  $z$ .

By Remark 3.3, we have

$$\mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x} = [\operatorname{Spec} R/G]$$

where  $R$  is a complete local ring and  $G$  acts freely away from a codimension 2 closed subscheme of  $\operatorname{Spec} R$ . Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . By Lemma 5.3,  $\widehat{\mathcal{O}}_{X,x}$  is isomorphic to the complete local ring of a pointed affine toric variety  $Y$  at its distinguished point  $y$ . Let  $\mathcal{Y}$  be the canonical stack over  $Y$ . By Remark 3.3, we have a cartesian diagram

$$\begin{array}{ccccc} \mathcal{X} & \longleftarrow & [\operatorname{Spec} R/G] & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & \operatorname{Spec} \widehat{\mathcal{O}}_{X,x} & \longrightarrow & Y \end{array}$$

By Corollary 5.8,  $\operatorname{Cl}(\widehat{\mathcal{O}}_{X,x}) \cong \operatorname{Cl}(Y)$ , so there is a divisor  $D'_i$  on  $Y$  whose class has the same image in  $\operatorname{Cl}(\widehat{\mathcal{O}}_{X,x})$  as that of  $D_i$ , and the images of the  $D_i$  generate  $\operatorname{Cl}(\widehat{\mathcal{O}}_{X,x})$  if and only if the  $D'_i$  generate  $\operatorname{Cl}(Y)$ . The residual representation of  $\bigoplus \mathcal{O}_{\mathcal{X}}(D_i)$  at  $z$  is faithful if and only if the residual representation of  $\bigoplus \mathcal{O}_{\mathcal{Y}}(D'_i)$  at  $z$  is faithful. Since  $y$  is the torus-invariant point of  $Y$ , this is equivalent to faithfulness of all residual representations  $\bigoplus \mathcal{O}_{\mathcal{Y}}(D'_i)$ , which by Lemma 3.5, holds exactly when the  $D'_i$  generate  $\operatorname{Cl}(Y)$ .  $\square$

*Example 5.9.* Consider the surface  $X$  cut out by  $xy = z^2(z+1)$ . The completed local ring at the singular point agrees with the completed local ring at the singular point of the toric variety  $xy = z^2$ , so by Corollary 5.8, we have that the formal local class group is  $\mathbb{Z}/2$  at the cone point, and trivial elsewhere.

We have the Weil divisor parameterized as  $(t(t+1), t, t)$ , or cut out by the ideal  $I = (x - y(y+1), y - z)$ . To see it is not Cartier, it suffices to note that it is not principal in the first infinitesimal neighborhood of the origin; its reduction is given by  $(x - y, y - z)$ .

By Theorem 5.1, if  $k$  is infinite and  $\operatorname{char}(k) \neq 2$ , we see that  $X$  is a quotient of a smooth variety by a finite diagonalizable group. In fact, since we are able to produce a single Weil divisor  $D$  which generates all formal local class groups, with  $2D$  Cartier, the proof of the theorem shows that there is a smooth variety  $U$  and a generic  $\mu_2$ -torsor  $U \rightarrow X$ , ramified exactly over  $D$ .  $\diamond$

*Example 5.10 (Strict toroidal embeddings are finite diagonalizable group quotients).* Recall that a *toroidal embedding* consists of a scheme  $X$  and an open subset  $W \subseteq X$  such that for every closed point  $x \in X$ , there exists a toric variety  $Y$  with torus  $T$ , a point  $y \in Y$  and an isomorphism of complete local rings  $\widehat{\mathcal{O}}_{X,x} \rightarrow \widehat{\mathcal{O}}_{Y,y}$  sending the ideal of  $X \setminus W$  to the ideal of  $Y \setminus T$ . The toroidal embedding  $W \subseteq X$  is said to be *strict* if every irreducible component of  $X \setminus W$  is a normal divisor.

Suppose  $W \subseteq X$  is a strict toroidal embedding, with  $X$  a quasi-projective variety with (automatically diagonalizable) quotient singularities over an infinite field. Let  $D_1, \dots, D_r$  be the irreducible components of  $X \setminus W$ . Given any point  $x \in X$ , choose a toric variety  $Y$ , a point  $y \in Y$ , and an isomorphism  $\widehat{\mathcal{O}}_{X,x} \rightarrow \widehat{\mathcal{O}}_{Y,y}$  as above. Since  $W \subseteq X$  is strict, the images of the  $D_i$  are linearly equivalent to the images of the torus-invariant divisors of  $Y$ , and therefore generate the formal local class groups by Corollary 5.8. Theorem 5.1 then shows that  $X$  is a quotient of a smooth scheme by a finite diagonalizable group.

This recovers the result [AK00, Lemma 7.1].  $\diamond$

## 5.1 Theorem 1.5: answering Question 2 negatively

In this subsection, we prove Theorem 1.5. Suppose  $X = \mathbb{P}(V)/A_5$  is of the form  $U/G$  for  $U$  a smooth variety and  $G$  a finite abelian group. At any point of  $U$ , the  $\operatorname{char}(k)$ -part  $P \subseteq G$  must

act by pseudoreflections, as the orders of the stabilizers of the canonical stack  $[\mathbb{P}(V)/A_5]$  have no  $\text{char}(k)$ -part (see Remark 3.2). It follows that  $U/P$  is smooth. Since  $X = (U/P)/(G/P)$ , to prove part (1) of Theorem 1.5, it suffices to show  $X$  is not the quotient of a smooth variety by a tame finite diagonalizable group.

The action of  $A_5$  on  $V$  induces a linearization of  $\mathcal{O}_{\mathbb{P}(V)}(1)$ . Let  $\mathcal{L}$  denote this linearized line bundle. Letting  $\alpha: A_5 \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$  be the action and  $p: A_5 \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$  be the projection, we let  $\phi: \alpha^*\mathcal{O}(1) \cong p^*\mathcal{O}(1)$  be the descent data corresponding to the linearization  $\mathcal{L}$ .

We show that  $\mathcal{L}^{\otimes n}$  is the unique  $A_5$ -linearization of  $\mathcal{O}(n)$ . Suppose  $\psi: \alpha^*\mathcal{O}(n) \cong p^*\mathcal{O}(n)$  is a choice of descent data corresponding to a linearization of  $\mathcal{O}(n)$ . Then  $\psi/\phi^n: A_5 \times \mathbb{P}(V) \rightarrow \mathbb{G}_m$  is given by a regular function  $\chi: A_5 \rightarrow \mathbb{G}_m$ , as regular functions on the components of  $A_5 \times \mathbb{P}(V)$  are constant. The cocycle conditions for  $\phi^n$  and  $\psi$  imply that  $\chi$  is a character of  $A_5$ , so it is trivial. Thus,  $\psi = \phi^n$ , as desired.

To show that  $X$  is not a quotient of a smooth variety by a tame finite diagonalizable group, by Theorem 5.1, it is enough to show that every line bundle on the canonical stack  $\mathcal{X} = [\mathbb{P}(V)/A_5]$  has trivial residual representations. If a line through the origin of  $V$  is fixed by an element  $g \in A_5$ , then  $g$  acts by rotation about that line. As a result, the stabilizers of points in  $\mathbb{P}(V)$  act trivially on the fibers of  $\mathcal{L}$ . Since  $\mathcal{O}(n)$  has a unique choice of linearization given by  $\mathcal{L}^n$ , the residual representations of every line bundle on  $\mathcal{X}$  are trivial. This completes the proof of part (1) of Theorem 1.5.

Let  $x$  be a singular point of  $X$ . By Remark 3.4, there is a correspondence between Weil divisors on  $X$  and line bundles on  $\mathcal{X}$ . Since every line bundle on  $\mathcal{X}$  has trivial residual representations, every Weil divisor on  $X$  has trivial image in  $\text{Cl}(\widehat{\mathcal{O}}_{X,x})$ . By Theorem 5.1 and Remark 5.2, any variety  $Y$  which contains a dense open subset isomorphic to a neighborhood of  $x$  in  $X$  cannot be of the form  $U/G$  with  $U$  smooth and  $G$  a tame finite diagonalizable group.

## 6 Appendix: a “bottom up” characterization of smooth DM stacks

In this appendix, we show that a smooth tame Deligne-Mumford stack can be recovered from its coarse space together with the ramification divisor of the coarse space morphism. Although we do not use this description in its full generality, we believe it is worth recording.

For an algebraic stack with quotient singularities  $\mathcal{Y}$ , let  $\mathcal{Y}^{\text{can}}$  denote the canonical stack over  $\mathcal{Y}$ .

**Theorem 6.1.** *Let  $\mathcal{X}$  be a smooth tame Deligne-Mumford stack with finite diagonal, trivial generic stabilizer, and suppose the coarse space  $X$  of  $\mathcal{X}$  has quasi-affine diagonal (e.g. if  $X$  is a scheme). Let  $D \subseteq X$  denote the ramification divisor of the coarse space map  $\pi: \mathcal{X} \rightarrow X$ , and  $\mathcal{D} \subseteq X^{\text{can}}$  denote the corresponding divisor in  $X^{\text{can}}$ . Let  $\sqrt{\mathcal{D}/X^{\text{can}}}$  denote the root stack of  $\mathcal{X}^{\text{can}}$  along the components of  $\mathcal{D}$  of order given by the orders of ramification of  $\pi$ . Then  $\sqrt{\mathcal{D}/\mathcal{X}^{\text{can}}}$  has quotient singularities and  $\pi$  factors as follows:*

$$\mathcal{X} \cong \sqrt{\mathcal{D}/X^{\text{can}}}^{\text{can}} \rightarrow \sqrt{\mathcal{D}/X^{\text{can}}} \rightarrow X^{\text{can}} \rightarrow X.$$

*Remark 6.2.* For  $\mathcal{X}$  with non-trivial generic stabilizer, Theorem 6.1 can be combined with [AOV08, Theorem A.1], which shows that  $\mathcal{X}$  is a gerbe over a smooth stack with trivial generic stabilizer.

**Lemma 6.3.** *Let  $\mathcal{U}$  be a Deligne-Mumford stack with finite diagonal and trivial generic stabilizer. If  $\mathcal{U}$  is étale over its coarse space, then  $\mathcal{U}$  is an algebraic space.*

*Proof.* It suffices to look étale locally on the coarse space of  $\mathcal{U}$ . We can therefore assume  $\mathcal{U} = [V/K]$ , where  $V$  is an algebraic space and  $K$  is the stabilizer of a geometric point  $v$  of  $V$ . Then  $V \rightarrow V/K$  is étale, and so  $K$  acts trivially on every jet space of  $v$  by the formal criterion for étaleness. On the other hand,  $K$  acts faithfully on some jet space of  $v$  by the proof of [EHKV01, Proposition 4.4] (smoothness is not needed). Thus,  $K$  is trivial, so  $\mathcal{U} = V$  is an algebraic space.  $\square$

**Corollary 6.4.** *Let  $\mathcal{U}$  and  $\mathcal{Y}$  be smooth Deligne-Mumford stacks with finite diagonal. Suppose a morphism  $g: \mathcal{U} \rightarrow \mathcal{Y}$  is birational and the ramification locus in  $\mathcal{U}$  is of codimension greater than 1. Then  $g$  is representable and étale.*

*Proof.* By Corollary 2.2(3), we can replace  $\mathcal{Y}$  by an étale cover, and so can assume  $\mathcal{Y} = Y$  is a scheme. Then  $\mathcal{U}$  has trivial generic stabilizer and  $g$  factors through the coarse space  $U$  of  $\mathcal{U}$ .

We show that the induced map  $\bar{g}: U \rightarrow Y$  is unramified in codimension 1. Let  $u \in U$  be a point of codimension 1 and let  $y \in Y$  be its image in  $Y$ . Since  $U$  is normal,  $\mathcal{O}_{U,u}$  is a discrete valuation ring. If  $V \rightarrow \mathcal{U}$  is an étale cover and  $v \in V$  is a point of codimension 1 which maps to  $u$ , then  $\mathcal{O}_{V,v}$  is unramified over  $\mathcal{O}_{Y,y}$ . Since ramification indices multiply in towers of discrete valuation rings, we see that  $\mathcal{O}_{U,u}$  is unramified over  $\mathcal{O}_{Y,y}$  as well.

Now we have  $U$  is normal,  $Y$  is smooth, and  $\bar{g}$  is dominant and unramified in codimension 1, so the purity of the branch locus theorem [SGA1, Exposé X, Theorem 3.1] shows that  $\bar{g}$  is étale. Similarly,  $V \rightarrow Y$  is a map between two smooth varieties which is dominant and unramified in codimension 1, so it is étale by purity. It follows that  $g$  is étale. Since  $g$  and  $\bar{g}$  are étale, the coarse space map  $\mathcal{U} \rightarrow U$  is étale, and hence an isomorphism by Lemma 6.3. This proves representability of  $g$ .  $\square$

**Lemma 6.5.** *Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of smooth Deligne-Mumford stacks with finite diagonal and trivial generic stabilizer. If  $f$  is unramified in codimension 1 and induces an isomorphism of coarse spaces, then  $f$  is an isomorphism.*

*Proof.* Let  $\pi: \mathcal{Y} \rightarrow Y$  be the coarse space of  $\mathcal{Y}$ . By assumption, the composition  $f \circ \pi: \mathcal{X} \rightarrow Y$  is a coarse space morphism. Since  $\mathcal{X}$  and  $\mathcal{Y}$  have trivial generic stabilizers and finite diagonal, these coarse space morphisms are birational, so  $f$  is birational. Since  $\mathcal{X}$  and  $\mathcal{Y}$  are proper and quasi-finite over  $Y$  [Con, Theorem 1.1],  $f$  is proper and quasi-finite. Since  $f$  is unramified in codimension 1 by construction, it is representable by Corollary 6.4. Zariski's Main Theorem [FMN10, Theorem C.1] then shows that  $f$  is an isomorphism.  $\square$

*Proof of Theorem 6.1.* By the universal property of canonical stacks (Remark 3.2), the coarse space morphism  $\pi: \mathcal{X} \rightarrow X$  factors through the canonical stack. Since  $X^{\text{can}} \rightarrow X$  is an isomorphism away from codimension 2, we have that  $\mathcal{D}$  is the ramification divisor of the map  $\tilde{\pi}: \mathcal{X} \rightarrow X^{\text{can}}$ .

Let  $\mathcal{D}_i$  be the irreducible components of  $\mathcal{D}$  and suppose  $\tilde{\pi}$  is ramified over  $\mathcal{D}_i$  with order  $e_i$ . Then  $\mathcal{Y} = \sqrt{\mathcal{D}/X^{\text{can}}}$  is obtained from  $X^{\text{can}}$  by taking the  $e_i$ -th root along  $\mathcal{D}_i$  for all  $i$ . By the universal property of root stacks, we get an induced morphism  $g: \mathcal{X} \rightarrow \mathcal{Y}$ . Note that each  $e_i$  must be relatively prime to the characteristic of  $k$ . This can be checked étale locally on  $X$ , so we may assume  $\mathcal{X}$  is a quotient of a smooth scheme by a finite group  $G$  which stabilizes a point. The ramification orders are then orders of subgroups of  $G$ , and the order of  $G$  is relatively prime to the characteristic of  $k$  since  $\mathcal{X}$  is assumed to be tame.

Let  $\mathcal{U} \subseteq X^{\text{can}}$  be the complement of the singular loci of the  $\mathcal{D}_i$  and the intersections of the  $\mathcal{D}_i$ . The restriction of  $\mathcal{D}$  to  $\mathcal{U}$  is a simple normal crossing divisor, so the open substack  $\mathcal{Y} \times_{X^{\text{can}}} \mathcal{U}$  of

$\mathcal{Y}$  is a smooth Deligne-Mumford stack by [FMN10, 1.3.b(3)]. Since  $\mathcal{Y} \rightarrow X^{\text{can}}$  ramifies with order  $e_i$  over  $\mathcal{D}_i$  and ramification orders multiply in towers of discrete valuation rings, we have that  $g$  is unramified in codimension 1. By Lemma 6.5,  $g$  restricts to an isomorphism of open substacks  $\mathcal{X} \times_{X^{\text{can}}} \mathcal{U} \rightarrow \mathcal{Y} \times_{X^{\text{can}}} \mathcal{U}$ . These open substacks have complements of codimension at least 2, so  $g$  is Stein (i.e.  $g_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{Y}}$ ).

Root stack and canonical stack morphisms have affine diagonal, so  $\mathcal{Y} \rightarrow X$  has affine diagonal. As  $\pi: \mathcal{X} \rightarrow X$  is a coarse space morphism of a tame stack, it is cohomologically affine [AOV08, Theorem 3.2]. By [Alp08, Proposition 3.14]  $g: \mathcal{X} \rightarrow \mathcal{Y}$  is cohomologically affine. As  $g$  is Stein and cohomologically affine, it is a good moduli space morphism, so it is universal for maps to algebraic spaces [Alp08, Theorem 6.6], so it is a relative coarse space morphism. Since  $\mathcal{X}$  is smooth, it follows that  $\mathcal{Y}$  has quotient singularities. Since  $g$  is an isomorphism away from codimension 2, it is a canonical stack morphism by Remark 3.2.  $\square$

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